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Effects of environmental parameters to total, quantum and classical correlations

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Abstract

We quantify the total, quantum and classical correlations with entropic measures, and quantitatively compare these correlations in a quantum system, as exemplified by a Heisenberg dimer which is subjected to the change of environmental parameters: temperature and nonuniform external field. Our results show that the quantum correlation may exceed the classical correlation at some nonzero temperatures, though the former is rather fragile than the latter under thermal fluctuation. The effect of the external field to the classical correlation is quite different from the quantum correlation.

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1. Introduction

Correlation effect plays an important role in physical phenomena. Many interesting properties of the quantum systems are attributed to the existence of the entanglement [1], which is intrinsically related to the superposition principle of quantum mechanics and the direct product structure of the Hilbert space [2, 3]. Entanglement is a kind of pure quantum correlation which does not exist in any classical systems, and regarded as a significant resource in quantum information processing, such as quantum teleportation, dense coding and quantum cryptography [4].

Due to the central role of the entanglement in quantum information, various issues on the entanglement have been studied intensively in the recent 15 years. Among these issues, the effects of environmental parameters (such as thermal fluctuation and external field) on the entanglement in quantum spin systems have attracted much attention [5–9]. Moreover, some interesting properties of the entanglement which are beyond the traditional physical intuition were found. For example, the thermal fluctuation can enhance the entanglement in some

special cases; the external field does not always suppress the entanglement [9]. Therefore, these studies shed new light on our understanding of the entanglement.

However, besides the quantum entanglement, a quantum system possesses the classical correlation [10, 11]. A simple example is the spin singlet state $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$, besides the entanglement with value 1, the state also has the classical correlation of value 1 (see section 2). Another example is the mixed state with the density matrix as $\rho = (|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)/2$. In this state, the quantum correlation between two spins is zero, while the classical correlation is 1. Therefore, some interesting questions arise. For example, what is the difference between the classical correlation and the quantum correlation in a realistic system? Is the classical correlation always larger than the quantum one? Why do we live in a classical world rather than a quantum world? etc. Answering these questions from the point of view of different correlations is our main motivation in this work.

Our paper is organized as follows. In section 2, we define the measurements of total, quantum and classical correlations. In section 3, we use a Heisenberg dimer which interacts with the thermal environment as an example to study the effects of the temperature, external fields and anisotropic interaction on the correlations. Finally, a summary is given in section 4.

2. Definitions and measures of bipartite correlations

2.1. Total correlation

In the quantum information theory [2], for two subsystems 1 and 2, the mutual information is defined as

$$S(1 : 2) = S(1) + S(2) - S(1 \cup 2), \quad (1)$$

where $S(i) = -\text{tr}(\rho_i \log_2 \rho_i)$, $i = 1, 2, 1 \cup 2$, is the entropy of the corresponding reduced density matrix. Since the entropy is used to quantify the physical resource (in unit of classical bit due to \log_2 in its expression) needed to store information of a system, the mutual entropy then measures the additional physical resource required if we store two subsystems respectively rather than store them together. Let us look at a very simple example: a two-qubit system in a singlet state $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. We have $S(1) = S(2) = 1$ and $S(1 \cup 2) = 0$, hence $S(1 : 2) = 2$. Obviously, there is no information in a given singlet state. However, each spin in this state is completely uncertain. So we need two bits to store them respectively. Here the mutual information is twice the entanglement, as measured by the von Neumann entropy of either subsystem. This is due to the reason that besides quantum correlation, the state has also classical correlation between the two subsystems. Therefore, the mutual information can be used to measure the total correlation between two subsystems. We will call the quantity $S(1 : 2)$ the ‘total correlation entropy’ or simply ‘total correlation’ hereafter.

2.2. Quantum correlation

The quantum correlation only exists in the quantum world, and usually is called entanglement. For a bipartite state, there are a few measures to quantify the entanglement of a general mixed state [12]. Among the measures, the entanglement of formation [13] is well known and an analytic formula for a two-qubit system is found [14]. Consider a density matrix ρ of two subsystems 1 and 2. There are infinite pure-state ensembles $\{|\psi_i\rangle, p_i\}$ of ρ , where p_i is the probability of $|\psi_i\rangle$, such that

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \quad (2)$$

For each pure state $|\psi_i\rangle$, the entanglement E is measured by the von Neumann entropy [15]. Then the entanglement of formation E_f of the density matrix ρ is the average entanglement of the pure states of the decomposition, minimized over all the possible ensembles:

$$E_f(\rho) = \min \sum_i p_i E(\psi_i). \quad (3)$$

(Note that if the system is in a pure state, E_f is just E .) For a mixed state, it is usually difficult to evaluate E_f . However, for a two-qubit system, it can be readily obtained from the concurrence of the system. Given the density matrix ρ of the pair qubits, the concurrence is given by [14]

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \quad (4)$$

where λ_i are the square roots of the eigenvalues of the operator

$$\varrho = \rho(\sigma_1^y \otimes \sigma_2^y)\rho^*(\sigma_1^y \otimes \sigma_2^y), \quad (5)$$

with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, σ_i^y are the normal Pauli operators, and ρ^* is the complex conjugate operator of ρ . The entanglement of formation can then be evaluated as [14]

$$E_f = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right), \quad (6)$$

$$h(x) = -x \log_2 x - (1 - x) \log_2(1 - x).$$

E_f is monotonically increasing and ranges from 0 to 1 as C goes from 0 to 1. $E_f = C = 0$ if the system is unentangled and $E_f = C = 1$ if it is maximally entangled. In fact, one can take the concurrence itself as a measurement of entanglement. Since the mutual information has the unit of bit, for comparison purposes, we will take the entanglement of formation instead of concurrence to be our measurement standard in this paper. We will call the quantity E_f the ‘quantum correlation entropy’ or simply ‘quantum correlation’ hereafter.

2.3. Classical correlation

The classical correlation of a bipartite system is defined in different scenarios [10, 11]. The measure defined in [10] reflects the effect of one party’s measurement on the other party’s state. The measure defined in [11] attempts to explain the total correlation coming from the quantum part and the classical part based on the distance concept of relative entropy. Both these two measures coincide in the case of pure states. Consider a pure state of a bipartite system, $|\psi\rangle = \sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle$ upon Schmidt decomposition. The quantum correlation actually defines the amount of immediate effect on one subsystem during the performing measurement on another subsystem. For the pure state, it is just the entropy of one subsystem. After the measurement, the density matrix becomes diagonal in the basis of the Schmidt decomposition. Then the classical correlation between these two subsystems corresponds to the maximum amount of change of uncertainty in one subsystem after knowing some new information of another subsystem through a classical channel. Such a correlation equals the entropy of one subsystem too. For the mixed state, the total correlation cannot neatly be divided into the quantum part and the classical part. These two parts are much more ‘entangled’ with each other.

In this paper, we follow the lines of [11]. That is, roughly speaking, the total correlation comes from the quantum part and the classical one. Intuitively, the quantum correlation is more flimsy than the classical one, and the classical correlation should be larger than the quantum correlation in the mixed state. An obvious instance is that for the separable state, there is no entanglement while classical correlation exists, in which case all the total correlation

comes from the classical part. Is it possible that the quantum correlation is larger than the classical part? For this purpose, we would like to adopt the quantum entanglement measure as large as possible. It is proved that all the reasonable entanglement measures are not larger than the entanglement of formation [16]. So, we take the entanglement of formation as the quantum correlation and the classical correlation is defined as the total correlation minus the quantum part. Before we discuss the main result, we argue that the total correlation minus the entanglement of formation is the non-erasable correlation under the constraint that entanglement is preserved. Recall that the entanglement of formation is originally proposed to describe the process of preparation of an entangled state under local operation and classical communication (LOCC). From equation (3), we can see that the entanglement of formation corresponds to a specified decomposition of the density matrix. In experiment, to have a given decomposition, one of experimentalists, say Alice, prepares the states $\{|\psi_i\rangle\}$ according to the probability distribution $\{p_i\}$. Therefore, the actual state that describes the initial state of the preparation process is

$$\bar{\rho} = \sum_i p_i |i\rangle\langle i| \otimes |\psi_i\rangle\langle\psi_i|, \quad (7)$$

where $\{|i\rangle\}$ are the flags, a set of orthogonal basis for Alice to distinguish $|\psi_i\rangle$. After compression, Alice sends the subsystem through an ideal quantum channel to the another experimentalist Bob who needs to know which one he receives because in general he cannot decompress the state without destroying the entanglement of the state. He requires the information to distinguish $|\psi_i\rangle$. Therefore, the flag states are also needed to be sent though this task does not require an ideal quantum channel. A classical channel is enough. In order to obtain the goal state ρ from the prepared state $\bar{\rho}$, both Alice and Bob are required to erase the flag memory which is used to store the information of the set of orthogonal basis. This procedure decreases the classical correlation but preserves the entanglement of formation. However any more information cannot be erased further or entanglement will be destroyed. Therefore, the remaining part of the correlation represents the non-erasable classical correlation between the two subsystems ρ under the preservation of quantum correlation, and is calculated as

$$S_C = S(1 : 2) - S_f. \quad (8)$$

In short, we argue that the mutual information is taken as the total correlation, the entanglement of formation is taken as the quantum correlation and the difference of them is the classical correlation in the meaning of non-erasable classical correlation. Especially for an arbitrary two-qubit system, the total, quantum and classical correlations can be easily calculated and we can compare the quantum correlation and the classical one quantitatively.

3. Environment's effects on correlations

In this section, we use the Heisenberg dimer as a prototype model to show the interesting behavior of the total, quantum and classical correlations under different environments. The model Hamiltonian reads

$$H = J \left[\frac{1-\gamma}{2} (\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + \frac{1+\gamma}{2} \sigma_1^z \sigma_2^z \right] + B_1 \sigma_1^z + B_2 \sigma_2^z, \quad (9)$$

where σ_i^α ($\alpha = x, y, z$) are the Pauli matrices, J is the strength of Heisenberg interaction, and B_1, B_2 are the external magnetic fields. For simplicity, we choose J as the energy unit. The parameter γ , which ranges from -1 to 1 , adjusts the anisotropic interaction.

3.1. The anisotropic Heisenberg model

We first consider the case of $B_1 = B_2 = 0$. The eigenstates and eigenvalues are

$$\begin{aligned}
 |\psi_0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), & E_0 &= \frac{-3 + \gamma}{2}, \\
 |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), & E_1 &= \frac{1 - 3\gamma}{2}, \\
 |\psi_2\rangle &= |\uparrow\uparrow\rangle, & E_2 &= \frac{1 + \gamma}{2}, \\
 |\psi_3\rangle &= |\downarrow\downarrow\rangle, & E_3 &= \frac{1 + \gamma}{2}.
 \end{aligned} \tag{10}$$

The ground state is $|\psi_0\rangle$ for $\gamma \neq 1$. At the thermal equilibrium, the density matrix of the system is

$$\rho(T) = \eta \begin{pmatrix} e^{-\frac{(1+\gamma)}{T}} & 0 & 0 & 0 \\ 0 & \cosh\frac{1-\gamma}{T} & -\sinh\frac{1-\gamma}{T} & 0 \\ 0 & -\sinh\frac{1-\gamma}{T} & \cosh\frac{1-\gamma}{T} & 0 \\ 0 & 0 & 0 & e^{-\frac{(1+\gamma)}{T}} \end{pmatrix}, \tag{11}$$

where the Boltzmann constant k_B is set as one and

$$\eta = \frac{1}{2[\cosh\frac{1-\gamma}{T} + e^{-(1+\gamma)/T}]}.$$

From the density matrix (11), the total, quantum and classical correlations can be calculated directly. The results are shown in figure 1. We see several interesting features from this figure. First, at a high temperature, all correlations approach zero. This is because the occupation probabilities of the unentangled states will be enhanced and the correlations will be diluted. The thermal fluctuation is the leading effects. Second, at a certain temperature range, the classical correlation exceeds the quantum correlation. This is obvious in cases of small γ (figures 1(a) and (b)). Third, when γ is close to 1 (figure 1(c)), the classical correlation may exhibit a local minimum at a low temperature. It is worth noting that the quantum correlation is smaller for a larger γ . The physical interpretation is that a larger γ corresponds to a more classical model, hence less amount of entanglement. When γ reaches 1, the quantum correlation vanishes for all temperatures (figure 1(d)). This is expected because when $\gamma = 1$ all eigenstates of the Hamiltonian are unentangled states; hence the resulting density matrix $\rho(T)$ is separable for all temperature.

The above features can be illustrated more clearly by defining a threshold temperature T_{th} , which is a function of γ . Above T_{th} , the quantum correlation completely vanishes. The quantum correlation reaches zero when the concurrence reaches zero. From equations (4), (5) and (11), the concurrence of the system is

$$C = \max \left\{ \frac{\sinh\frac{1-\gamma}{T} - e^{-(1+\gamma)/T}}{\cosh\frac{1-\gamma}{T} + e^{-(1+\gamma)/T}}, 0 \right\}.$$

The $C = 0$ requires $\sinh\frac{1-\gamma}{T} \leq e^{-(1+\gamma)/T}$. Then the threshold temperature T_{th} should satisfy

$$\gamma = \frac{T_{\text{th}}}{2} \ln(e^{2/T_{\text{th}}} - 2). \tag{12}$$

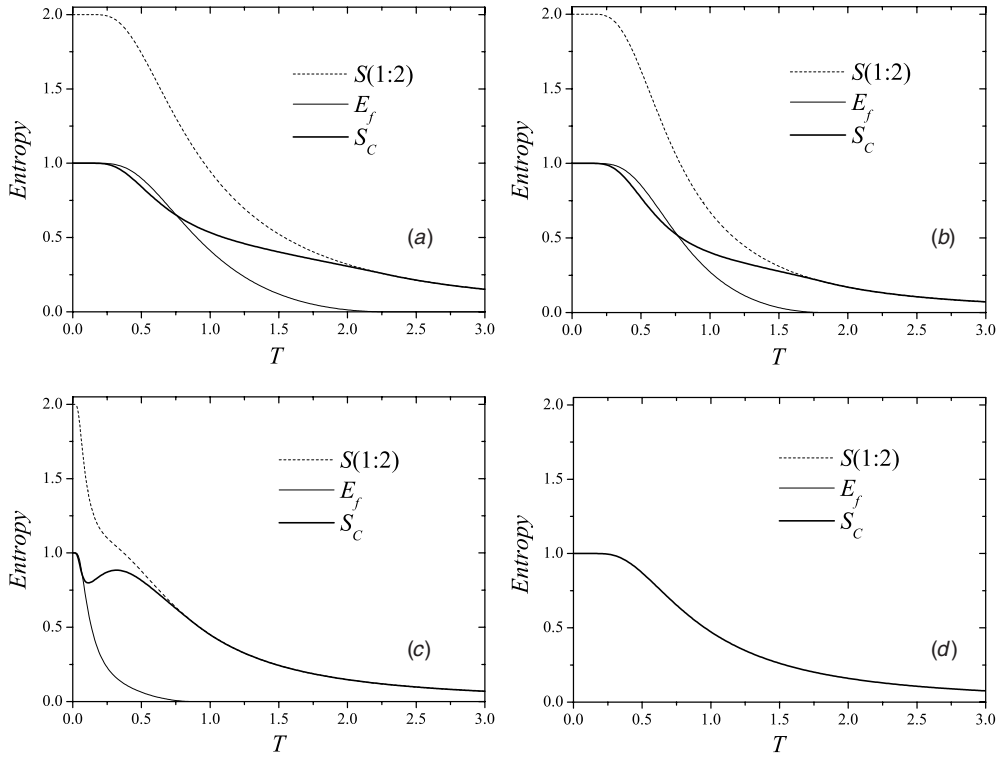


Figure 1. The total correlation $S(1 : 2)$, quantum correlation E_f and classical correlation S_C versus temperature T , in the anisotropic Heisenberg dimer with different values of parameter γ . In figure (d), the $E_f = 0$ and the curves of $S(1 : 2)$ and S_C are overlap. (a) $\gamma = -1$, (b) $\gamma = 0$, (c) $\gamma = 0.9$ and (d) $\gamma = 1$.

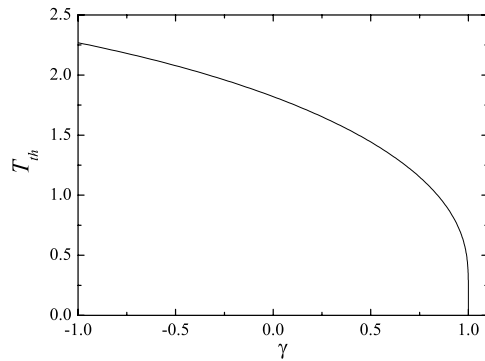


Figure 2. The threshold temperature T_{th} versus anisotropic parameter γ in the Heisenberg dimer. The quantum correlation of the system vanishes if $T > T_{th}$.

The plot of T_{th} versus γ is shown in figure 2. It is obvious that T_{th} drops when γ increases. From figure 2, we can divide the whole plane into two regions. Below the line of T_{th} , the system has both quantum and classical correlations, while above the line, the system has only classical correlation.

3.2. The XY model with a nonuniform magnetic field

Now we investigate the correlation effects of the external magnetic fields. We only consider the case $\gamma = -1$ and other γ can be obtained similarly. Then the eigenstates and eigenvalues of the Hamiltonian are

$$\begin{aligned} |\Psi_1\rangle &= |\uparrow\uparrow\rangle, & E_1 &= B_1 + B_2, \\ |\Psi_2\rangle &= |\downarrow\downarrow\rangle, & E_2 &= -(B_1 + B_2), \\ |\Psi_{\pm}\rangle &= \frac{1}{N_{\pm}} \left[\frac{(B_1 - B_2) \pm \sqrt{D}}{2} |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right], \\ E_{\pm} &= \pm\sqrt{D}, \end{aligned} \quad (13)$$

where $D = (B_1 - B_2)^2 + 4$ and N_{\pm} are the normalization factors. The thermal equilibrium state can be described by the density matrix

$$\rho(T) = \frac{1}{Z} \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & b - c & -s & 0 \\ 0 & -s & b + c & 0 \\ 0 & 0 & 0 & d^{-1} \end{pmatrix}, \quad (14)$$

where $Z = 2\{\cosh[(B_1 + B_2)/T] + \cosh(\sqrt{D}/T)\}$, $b = \cosh(\sqrt{D}/T)$, $c = \sinh(\sqrt{D}/T)$, $(B_1 - B_2)/\sqrt{D}$, $s = 2\sinh(\sqrt{D}/T)/\sqrt{D}$ and $d = \exp[-(B_1 + B_2)/T]$.

We first study the correlations under uniform magnetic fields at finite temperatures. The results are shown in figures 3(a)–(d). Clearly, if B_1 is small, the ground state $|\Psi_{-}\rangle$ is a superposition of two antiferromagnetic bases, and is entangled. If the system is subject to a thermal environment, the contribution from the other eigenstates (two of them are separable) will suppress both the quantum and classical correlations. However, if B_1 is large enough, the ground state becomes $|\Psi_2\rangle$, which is fully polarized and not entangled. The classical correlation, whose value is equal to the quantum one for a pure state, is also zero. The thermal fluctuation, as can be seen from figures 3(c) and (d), increases both the quantum and classical correlations at low temperatures. Moreover, an interesting observation is that there exists a range where the quantum correlation exceeds the classical one. In addition, the larger the external field the smaller the quantum correlation. It is because a large-field setting corresponds to a more classical model. It may be interesting to note that the threshold temperature of the quantum correlation is independent of the field [9]. All three correlations approach or equal zero at high temperatures.

If the directions of the two external fields are opposite to each other and the strengths are the same, we find that all the total, classical and quantum correlations show comparatively gentle changes against the temperature and fields (figures 4(a) and (b)). The figure is not difficult to interpret, as we argued for the case of $B_1 = B_2$. The main difference is that a larger B_1 here may lead to a higher threshold temperature T_{th} . In order to see the role of the nonuniform field, we show three correlations against fields at some fixed temperatures in figure 5. (The results of quantum correlation, which have already been obtained by Sun *et al* [9], are also presented for comparison.) At a low temperature, we note that the three correlations are sharply peaked at zero fields. They decay rapidly with the increasing fields if the fields have the same direction, while decay comparatively slowly if the fields have opposite directions. This means that the correlation effects can be adjusted by the uniform fields. At some higher temperature, the peak of the quantum correlation splits into two in the region $B_1 B_2 < 0$ (figure 5(b)). Therefore, the nonuniform fields may enhance the quantum

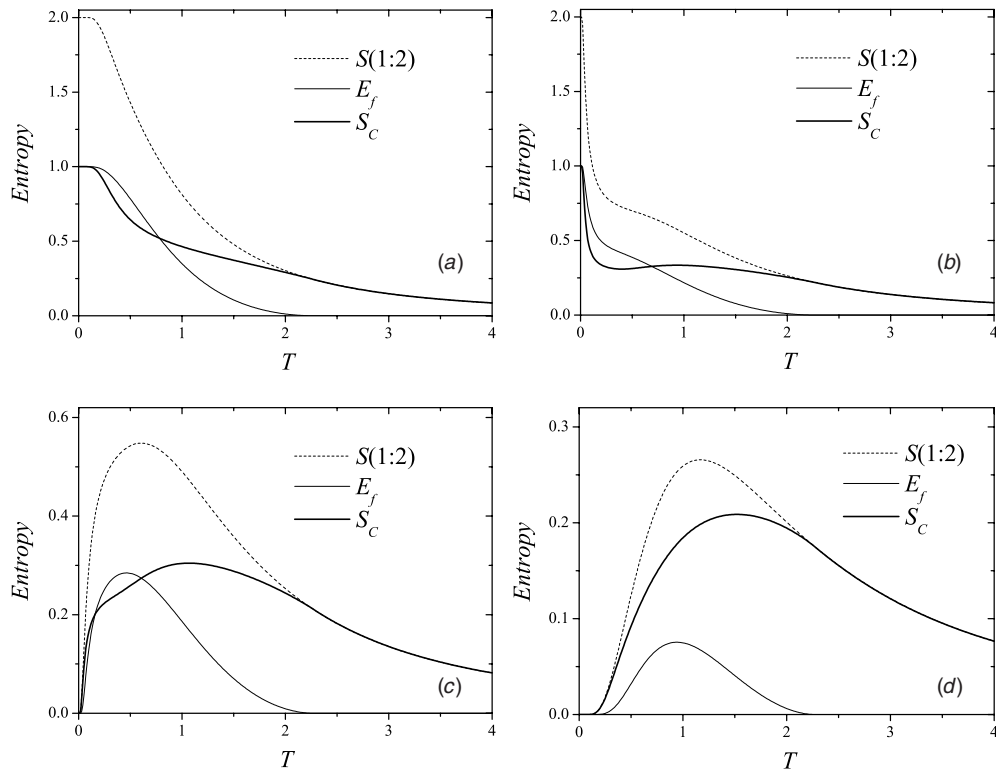


Figure 3. The total correlation $S(1 : 2)$, quantum correlation E_f and classical correlation S_C versus temperature T , in the XY model under a uniform magnetic field $B_1 = B_2$. Varying the strength of the fields, four typical cases are shown: (a) $B_1 = B_2 = 0.5$, (b) $B_1 = B_2 = 0.95$, (c) $B_1 = B_2 = 1.05$ and (d) $B_1 = B_2 = 1.5$.

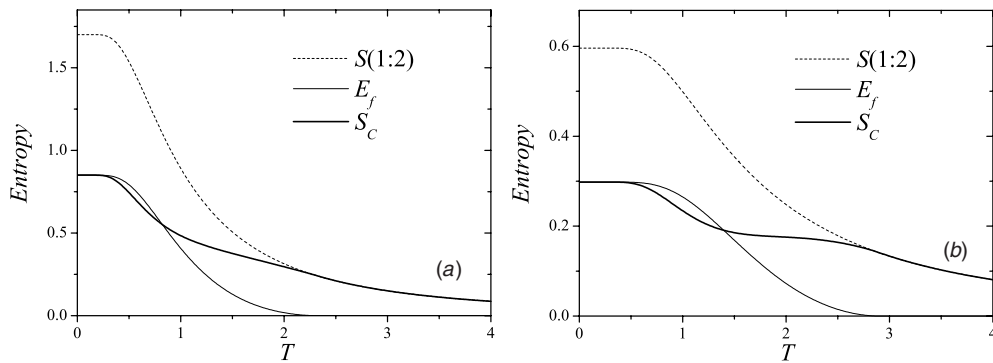


Figure 4. The total correlation $S(1 : 2)$, quantum correlation E_f and classical correlation S_C versus temperature T , in the XY model under a nonuniform field: (a) $B_1 = 0.5$, $B_2 = -0.5$ and (b) $B_1 = 2$, $B_2 = -2$.

correlation, while the uniform fields always destruct it. At a very high temperature, the peaks are completely separated as shown in figure 5(c). A region with zero quantum correlation appears between the peaks. This implies that the nonuniform field can be used as a switch

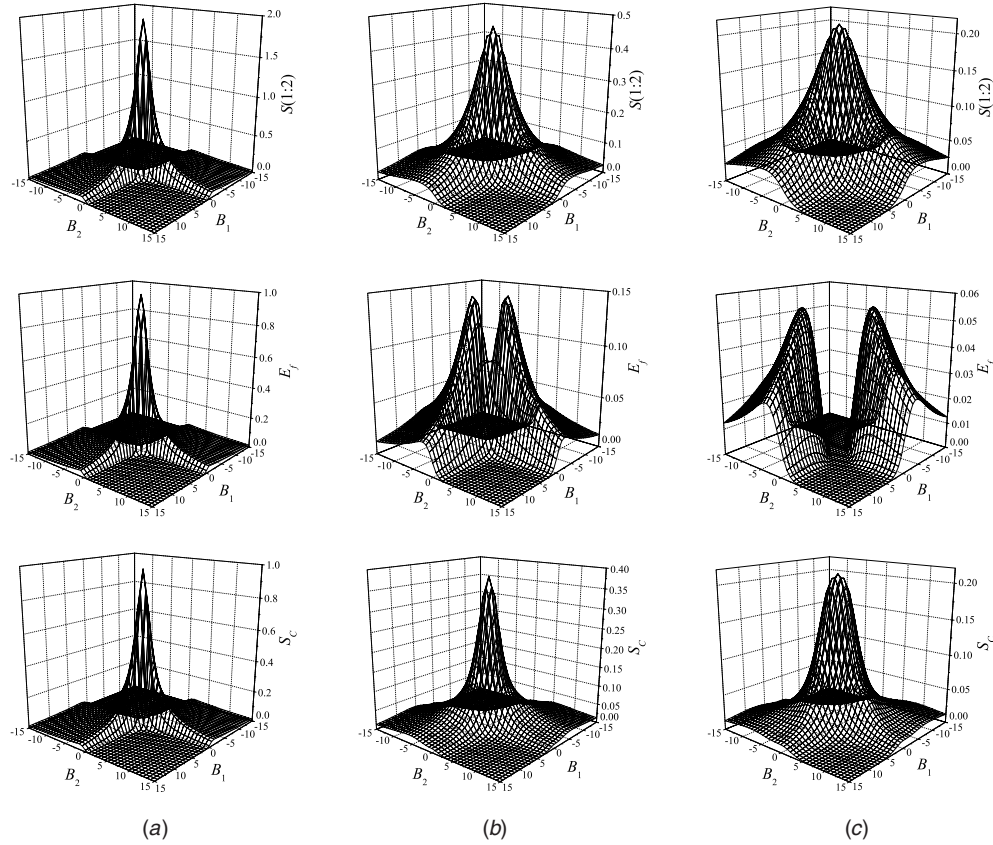


Figure 5. (From top to bottom) the total correlation $S(1 : 2)$, quantum correlation E_f and classical correlation S_C versus the external magnetic fields B_1 and B_2 at different fixed temperatures T in the XY model. Three typical cases are shown in different columns: (a) $T = 0.3$, (b) $T = 1.6$ and (c) $T = 2.5$.

to turn on and off the quantum correlation [9]. Meanwhile, unlike the quantum correlation, the classical correlation always decreases with the increasing external magnetic fields, which means that the external magnetic fields have different effects to the quantum correlation and to the classical correlation.

4. Discussions and summary

In this paper, we provide quantification of the total, quantum and classical correlations in a general bipartite system. In order to see their properties in a realistic system, we study them in an anisotropic Heisenberg model at finite temperatures. We find that the quantum correlation always decreases with increasing temperature, while the classical one may increase in some temperature range. More interestingly, the classical correlation is not always larger than the quantum one, which actually is beyond the general physical intuition [11]. We also investigate the three correlations in the XY model under a nonuniform magnetic field. We find that the fields may enhance the quantum correlation, which is very different from the effect of fields to the classical correlation. In short, our results imply that the environmental

parameters (temperature, magnetic fields) demonstrate obviously different effects to the quantum correlation and the classical correlation.

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